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The Kadeč–Pełczyński Dichotomy for the Predual of a von Neumann Algebra

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We first improve one of Akemann's criteria of weak compactness in the predual of a von Neumann algebra (denoted $\mathfrak{M}_*$). We then show that a non-reflexive subspace of $\mathfrak{M}_*$ contains a subspace isomorphic to l_1 which is complemented in $\mathfrak{M}_*$.

1. INTRODUCTION

In this article, all Banach spaces are considered over the complex field and the term subspace will always mean closed subspace.

Let $\mathfrak{h}$ be a separable Hilbert space and $L(\mathfrak{h})$ be the space of all bounded linear operators from $\mathfrak{h}$ to itself. A *von Neumann algebra* always means a self-adjoint subalgebra of $L(\mathfrak{h})$ which is closed for the strong (or weak) operator topology. It is well known that a von Neumann algebra, $\mathfrak{M}$, has a unique (up to isometric isomorphism) predual, usually denoted $\mathfrak{M}_*$. Let l_1 denote the space of absolutely summable sequences with its usual norm. Our main result is the following:

THEOREM 1. *If X is a non-reflexive subspace of $\mathfrak{M}_*$ then X contains a subspace isomorphic to l_1 which is complemented in $\mathfrak{M}_*$.*

The set $\mathfrak{M} = L^\infty(0, 1)$ identified with the multiplication operators on $L^2(0, 1)$ is an abelian von Neumann subalgebra of $L(L^2(0, 1))$. Kadeč and Pełczyński [7] proved theorem 1 for the case $\mathfrak{M}_* = L^1(0, 1)$.

If $\mathfrak{M} = L(\mathfrak{h})$ then $\mathfrak{M}_* = L_1(\mathfrak{h})$ is the space of all trace class operators on $\mathfrak{h}$ with the trace norm. Holub [6] has shown that a subspace of $\mathfrak{M}_*$ is either isomorphic to $\mathfrak{h}$ or it contains a copy of l_1. In conjunction with

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theorem 1, we get that every non-Hilbertian subspace of $L_1(\mathfrak{h})$ contains a complemented copy of l_1.

Now the principal argument used to produce a copy of l_1 in a non-reflexive subspace of the predual of a von Neumann algebra, is a sharpening of a characterization of weak compactness in $\mathfrak{M}_*$ due to Akemann [1]. The following result is the construction of what is commonly called a "sliding hump."

THEOREM 2. *Let K be a bounded subset of $\mathfrak{M}_*$. Then K is not weakly relatively compact iff for every $\varepsilon > 0$, there exist a constant $\alpha > 0$, a sequence of mutually orthogonal projections $\{p_n\}$ in $\mathfrak{M}$ and a sequence $\{\phi_n\}$ in K such that*

$$|\phi_n(p_n)| > \alpha$$

and

$$|\phi_n| \left(\sum_{k \neq n} p_k \right) < \varepsilon$$

$\forall n \in \mathbb{N}$.

2. PRELIMINARIES

The first part of this section introduces the concepts and results needed from Banach space theory. The interested reader is referred to [4]. The preliminaries from the theory of von Neumann algebras form the second part of this section; [8] and [9] are standard references on the subject. Section 3 contains the proofs of the two theorems announced in the introduction.

Let X be a Banach space with norm denoted by $\|\cdot\|$. Then X is *reflexive* if $B_X = \{x : x \in X, \|x\| \leqslant 1\}$ is a $\sigma(X, X^*)$-compact subset of X. Let Y be a subspace of X with inclusion i. Then Y is a *complemented* subspace of X if there exists a bounded linear operator P: $X \to Y$ such that the composition of P with i, $P \circ i$: $Y \to Y$, is the identity on Y.

A sequence $\{x_n\}$ is called a *(Schauder) basis* for X if for each x in X there exists a unique sequence of complex numbers (λ_n) such that $x = \lim_{N \to \infty} \sum_{n=1}^{N} \lambda_n x_n$. More generally, $\{x_n\}$ is called a *basic sequence* when it is a basis of its closed linear span. For each n in $\mathbb{N}$, let e_n denote the sequence in l_1 whose nth term is 1 while all the others are 0. Then $\{e_n\}$ is a basis for l_1 called the *unit vector basis* of l_1.

Let $\{x_n\}$ be a basic sequence in X and $\{y_n\}$ be a basic sequence in Y. We say that $\{x_n\}$ and $\{y_n\}$ are *equivalent* if, given any sequence of complex numbers (λ_n), the convergence of $\sum \lambda_n x_n$ is equivalent to the convergence of $\sum \lambda_n y_n$. It is a consequence of the closed graph theorem that $\{x_n\}$ and $\{y_n\}$ are equivalent iff the application T which maps x_n to y_n for each n in $\mathbb{N}$ extends to an isomorphism of Banach spaces from the closed linear span of $\{x_n\}$ to the closed linear span of $\{y_n\}$. If $C = \|T\| \, \|T^{-1}\|$, then $\{x_n\}$ and $\{y_n\}$ are said to be C-equivalent.

Now let us state a remarkable property of l_1 in the predual of a von Neumann algebra. This result is due to Arazy [2] and was first proved for the case $\mathfrak{M}_* = L^1(0, 1)$ by Dor [5].

THEOREM 3. *There exists a constant C_0 with $1 < C_0 < +\infty$ having the property: If $\{x_n\}$ is a sequence in the predual of a von Neumann algebra $\mathfrak{M}$ which is C-equivalent to the unit vector basis of l_1 with $C < C_0$, then the isomorphic copy of l_1 which is spanned by $\{x_n\}$ is complemented in $\mathfrak{M}_*$.*

An element y of a von Neumann algebra $\mathfrak{M}$ is positive if there exists an element a in $\mathfrak{M}$ such that $y = a^*a$. The positive elements form a cone and as such define a linear order on $\mathfrak{M}$. An element in the real subspace spanned by the positive elements is said to be *hermitian*. Let $\{x_\alpha\}_{\alpha \in A}$ be a bounded increasing net of hermitian elements in $\mathfrak{M}$. A continuous linear functional ϕ on $\mathfrak{M}$ is said to be *normal* if $\phi(\sup\{x_\alpha\}_{\alpha \in A}) = \sup\{\phi(x_\alpha)\}_{\alpha \in A}$. A *projection* p in $\mathfrak{M}$ is a hermitian element such that $p = p^2$. Two projections p and q are said to be *orthogonal* if $pq = qp = 0$. If $\{p_n\}_{n \in \mathbb{N}}$ is a sequence of mutually orthogonal projections then $\{\sum_{n=1}^{N} p_n\}_N$ forms a bounded and increasing sequence in $\mathfrak{M}$ so that $\phi(\sum_{n=1}^{\infty} p_n) = \sum_{n=1}^{\infty} \phi(p_n)$ whenever ϕ is normal. The Banach space of all the normal functionals on $\mathfrak{M}$ is $\mathfrak{M}_*$, the predual of $\mathfrak{M}$. The following theorem due to Akemann [1] is a characterization of weak-compactness for bounded subsets of $\mathfrak{M}_*$.

THEOREM 4. *A bounded subset K of $\mathfrak{M}_*$ is relatively weakly compact if and only if for any sequence $\{p_n\}$ of mutually orthogonal projections in $\mathfrak{M}$, we have $\lim_n \phi(p_n) = 0$ uniformly in $\phi \in K$.*

A normal functional ϕ is said to be *positive* if $\phi(a^*a) \geqslant 0$ for all a in $\mathfrak{M}$. If ϕ is positive then $\phi(\mathbf{1}) = \|\phi\|$ where $\mathbf{1}$ is the unit in $\mathfrak{M}$. For each ϕ in $\mathfrak{M}_*$ there exists a unique positive functional $|\phi|$, also in $\mathfrak{M}_*$, such that $\||\phi|\| = \|\phi\|$ and

$$|\phi(a)|^2 \leqslant \|\phi\| \, |\phi| \, (a^*a) \tag{2.1}$$

for all a in $\mathfrak{M}$. A partial isometry in $\mathfrak{M}$ is an element u having the property that u^*u is a projection. Non-zero partial isometries, like projections, have norm equal to one. If ϕ and $|\phi|$ are as above then there exists a partial isometry u in $\mathfrak{M}$ such that $|\phi| \, (\cdot) = \phi(u \cdot)$ and $\phi(\cdot) = |\phi| \, (u^* \cdot)$. The couple

$|\phi|$ and u is called the *polar decomposition* of ϕ. Note that $\mathfrak{M}_*$ is a left and right module over $\mathfrak{M}$ for the operations $a\phi(\cdot)=\phi(a\cdot)$ and $\phi a(\cdot)=\phi(\cdot a)$ for a in $\mathfrak{M}$ and ϕ in $\mathfrak{M}_*$. The polar decomposition of ϕ can therefore be written $\phi=u^*|\phi|$. The last basic inequality we will need states that for a and ϕ as above and for all positive y in $\mathfrak{M}$,

$$|\phi(ay)| \leqslant \|a\| \, |\phi| \, (y). \tag{2.2}$$

3. The Proofs of the Two Theorems

Since Theorem 2 is instrumental in proving Theorem 1, we start with its proof.

Proof of Theorem 2. Let K be a bounded subset of the predual $\mathfrak{M}_*$ of a von Neumann algebra $\mathfrak{M}$. We will assume without loss of generality that all the elements of K have norm less than or equal to 1. Suppose that K is not relatively weakly compact. By Akemann's criterion above (Theorem 4), there exists a constant $\alpha>0$, a sequence of orthogonal projections $\{q_n\}$ in $\mathfrak{M}$ and a sequence $\{\psi_n\}$ in K such that

$$|\psi_n(q_n)| > \alpha \qquad \forall n \in \mathbb{N}.$$

For each n in $\mathbb{N}$, define the following functions on the subsets of $\mathbb{N}$:

$$\mu_n(\Delta) := |\psi_n| \left(\sum_{j\in\Delta} q_j \right) = \sum_{j\in\Delta} |\psi_n| \, (q_j) \qquad \forall \Delta \in 2^{\mathbb{N}}$$

Note that since each $|\psi_n|$ is positive, each μ_n is positive. Also, for Δ_1 and Δ_2 disjoint subsets of $\mathbb{N}$, we have that

$$\begin{aligned} \mu_n(\Delta_1 \cup \Delta_2) &= \sum_{j\in\Delta_1\cup\Delta_2} |\psi_n| \, (q_j) = \sum_{j\in\Delta_1} |\psi_n| \, (q_j) + \sum_{j\in\Delta_2} |\psi_n| \, (q_j) \\ &= \mu_n(\Delta_1) + \mu_n(\Delta_2), \end{aligned}$$

which shows that μ_n is additive for every n in $\mathbb{N}$. Moreover, we have for every Δ in $2^{\mathbb{N}}$,

$$\mu_n(\Delta) = |\psi_n| \left(\sum_{j\in\Delta} q_j \right) \leqslant |\psi_n| \, (\mathbf{1}) \leqslant 1 \qquad \forall n \in \mathbb{N}.$$

Thus $\{\mu_n\}$ is a uniformly bounded sequence of positive and additive functions on $2^{\mathbb{N}}$.

Let $\varepsilon>0$. Apply Rosenthal's lemma [4, p. 82] in order to get an increasing sequence $\{k_n\}$ of positive integers such that

$$\mu_{k_n}\left(\bigcup_{j\neq n}\{k_j\}\right)<\varepsilon \qquad \forall n\in\mathbb{N}.$$

Now let $\phi_n=\psi_{k_n}$ and $p_n=q_{k_n}$. Then $\{p_n\}$ is a sequence of mutually orthogonal projections in $\mathfrak{M}$, $\{\phi_n\}$ a sequence of elements belonging to K such that:

$$|\phi_n(p_n)|>\alpha \qquad \text{and} \qquad |\phi_n|\left(\sum_{j\neq n} p_j\right)<\varepsilon \qquad \forall n\in\mathbb{N}.$$

This proves Theorem 2.

Remark. We have just proved the following consequence of Rosenthal's lemma:

PROPOSITION 5. *Let $\{\phi_n\}$ be a bounded sequence in $\mathfrak{M}_*$. For any $\varepsilon>0$ and any sequence of orthogonal projections $\{p_n\}$, there exists an increasing sequence of positive integers $\{k_n\}$ such that*

$$|\phi_{k_n}|\left(\sum_{j\neq n} p_{k_j}\right)<\varepsilon \qquad \forall n\in\mathbb{N}.$$

Proof of Theorem 1. Let X be a subspace of $\mathfrak{M}_*$ which is not reflexive. This implies that $K=B_X$ is not $\sigma(X,X^*)$-compact. It is a simple consequence of the Hahn–Banach theorem that K cannot be $\sigma(X,\mathfrak{M})$-compact either. Using Theorem 2, we get for each $\varepsilon>0$ a constant $\alpha>0$, a sequence of mutually orthogonal projections $\{p_n\}$ in $\mathfrak{M}$ and a sequence $\{\phi_n\}$ in K such that $|\phi_n(p_n)|>\alpha$ and $|\phi_n|\,(\sum_{k\neq n}(p_k))<\varepsilon$ $\forall n\in\mathbb{N}$. Let $p=\sum_j p_j$. Now for N in $\mathbb{N}$ big enough, select a subsequence $\{\phi_{k_n}\}$ such that

$$\frac{r}{2^N}<\|p\phi_{k_n}p\|\leqslant\frac{r+1}{2^N}$$

for a certain r: $1\leqslant r\leqslant 2^N-1$. Define $\chi_n:=(2^N/r)\,p\phi_{k_n}p$ and $q_n:=p_{k_n}$ for all n in $\mathbb{N}$. Since

$$1\leqslant\|\chi_n\|\leqslant\frac{r+1}{r}, \qquad \forall n\in\mathbb{N}.$$

we clearly get for each $(\lambda_n)\in l_1$,

$$\left\|\sum_n \lambda_n\chi_n\right\|\leqslant\frac{r+1}{r}\,\|(\lambda_n)\|_1.$$

Using the other side of the inequality we get

$$|\chi_n|\,(q_n) = |\chi_n|\,(p) - |\chi_n|\,(p - q_n) \geqslant 1 - \frac{2^N}{r}\,\varepsilon.$$

For each n in $\mathbb{N}$, let u_n be the partial isometry appearing in the polar decomposition of each χ_n, i.e., $|\chi_n|\,(\cdot) = \chi_n(u_n \cdot)$ for each n. Thus

$$\begin{aligned}|\chi_n(q_n u_n q_n)| &\geqslant \chi_n(u_n q_n) - |\chi_n|\,((p - q_n) u_n q_n)\\ &\geqslant |\chi_n|\,(q_n) - |\chi_n|\,(p - q_n) \geqslant 1 - \frac{2^{N+1}}{r}\,\varepsilon.\end{aligned}$$

Let $u = \sum_{k=1}^{\infty} (\bar{\lambda}_k/|\lambda_k|) q_k u_k q_k$. This series is convergent in the strong operator topology by the uniform boundedness principle and so is $\sum_{k=1}^{\infty} (\lambda_k/|\lambda_k|) q_k u_k^* q_k = u^*$. For a fixed n in $\mathbb{N}$,

$$\begin{aligned}\left|\chi_n\left(\sum_{k \neq n} \frac{\bar{\lambda}}{|\lambda_k|}\, q_k u_k q_k\right)\right|^2 &\leqslant \frac{r+1}{r}\,|\chi_n|\left(\sum_{k \neq n} q_k u_k^* q_k u_k q_k\right)\\ &\leqslant \frac{r+1}{r} \sum_{k \neq n} |\chi_n|\,(q_k)\\ &\leqslant \frac{r+1}{r}\,\frac{2^N}{r}\,\varepsilon,\end{aligned}$$

where the first inequality uses (2.1) and the second (2.2).

Therefore,

$$\begin{aligned}\left\|\sum_n \lambda_n \chi_n\right\| &\geqslant \left|\sum_n \lambda_n \chi_n(u)\right|\\ &= \left|\sum_n \left(|\lambda_n|\, \chi_n(q_n u_n q_n) + \lambda_n \chi_n\left(\sum_{k \neq n} \frac{\bar{\lambda}_k}{|\lambda_k|}\, q_k u_k q_k\right)\right)\right|\\ &\geqslant \left(1 - \frac{2^{N+1}\sqrt{\varepsilon} - \sqrt{2^N(r+1)}}{r}\,\sqrt{\varepsilon}\right) \sum_n |\lambda_n|,\end{aligned}$$

which shows that $\{\chi_n\}$ is equivalent to the unit vector basis of l_1 with equivalence constant

$$C = \frac{r+1}{r}\left(1 - \frac{2^{N+1}\sqrt{\varepsilon} - \sqrt{2^N(r+1)}}{r}\,\sqrt{\varepsilon}\right)^{-1}.$$

If we choose N big enough and ε small enough we get a constant $C < C_0$, where C_0 is the universal constant in the theorem of Arazy (Theorem 3).

Thus we have that the closed linear span of $\{\chi_n\}$ is an isomorphic copy of l_1 which is complemented in $\mathfrak{M}_*$. Since the map $R: \mathfrak{M}_* \to p\mathfrak{M}_* p$ given by $\phi \mapsto p\phi p$ is linear, bounded, surjective and such that $R((2^N/r)\phi_{k_n}) = \chi_n$, we have that $\overline{\text{span}}\{\phi_{k_n}\}$ is a subspace of X which is complemented in $\mathfrak{M}_*$ and, of course, still isomorphic to l_1 (this "pullback" argument is essentially the proof of Sobczyk's theorem [4, p. 72]).

COROLLARY 6. *If K is a bounded subset of $\mathfrak{M}_*$ which is not weakly relatively compact then K contains a sequence equivalent to the unit vector basis of l_1.*

Finally, as announced in the Introduction, the following result is an improvement on a theorem of Holub [6].

COROLLARY 7. *If X is a subspace of $L(\mathfrak{h})$ then X is either isomorphic to $\mathfrak{h}$, in which case it is complemented in $L(H)$, or it contains an isomorphic copy of l_1 which is complemented in $L(\mathfrak{h})$.*

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